

Computation and Visualization of the Equioptic for Two Ellipses and Equioptic for Infinitely Many Circles and Spheres

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Abstract

In this paper, we extend the original locus problem discussed in [1]. We are given two conic curves, denoted by E_1 and E_2 , respectively. Point A is outside these two conic curves. We draw tangents from A to the two curves, E_1 and E_2 , respectively. An equioptic curve for two ellipses is the locus of points A from which the two ellipses subtend equal viewing angles. In this paper, we shall see how a uniform scaling, shiftings, and uniform shear will affect the quasi-locus discussed in this paper. We further explore the equioptics for infinitely many circles in 2D within an evenly divided circle in a given circle. We extend this to corresponding 3D to visualize the equioptics for infinitely many spheres within an evenly divided cone in a given sphere. It is clear that without the help of a computer algebra system, it is impossible to compute and visualize the equioptics described in this paper.

We are given two disconnected circles, C_1 , and C_2 centered at O_1 and O_2 , respectively. Point $A = (x, y) \in \mathbb{R}^2$ is outside both circles, and we construct two pairs of tangent lines from A toward circles C_1 and C_2 , respectively. We label those points of tangency as B, C, D , and E , respectively; (see Figure 1). If $\angle BAC = \angle DAE$, we call point A an equal viewpoint.

Question: Find the locus of equal viewpoint A .

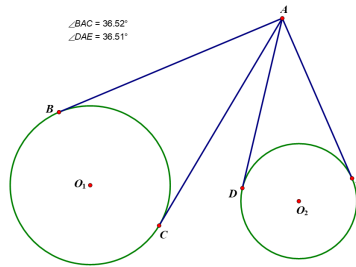


Figure 1. Equal viewpoint A .

Remarks:

An equioptic curve is exactly the *true locus* of equal viewpoints for two given smooth (continuously differentiable) curves, which we use in this paper. It is trivial, by its definition, that the equal viewpoint A has to be outside the two respective circles.

1 Obtaining the first stage quasi-equioptic equation

The existence of a quasi-equioptic for two ellipses is that neither ellipse is contained within the other. Unless otherwise specified, the ellipses we discuss in this paper are non-degenerate cases.

In [1], we start with a simple case with the following two respective equations of ellipses, E_1 (1) and E_2 (2) respectively:

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \quad (1)$$

and

$$\frac{(x - c)^2}{a_2^2} + \frac{(y - d)^2}{b_2^2} = 1. \quad (2)$$

We assume a_1, b_1, a_2 , and $b_2 > 0$. We set up the tangent line equation with slope k_1 for E_1 , which passes through the equal viewpoint $A(x_0, y_0)$ as follows

$$y - y_0 - k_1(x - x_0) = 0. \quad (3)$$

Similarly, we set up the tangent line equation with slope k_2 for E_2 , which passes through the equal viewpoint $A(x_0, y_0)$ as follow

$$y - y_0 - k_2(x - x_0) = 0. \quad (4)$$

1. We substitute first tangent line equation (3) into the first ellipse (1). After simplifications, we obtain the following:

$$a_1^2 k_1^2 x^2 - 2a_1^2 k_1^2 x x_0 + a_1^2 k_1^2 x_0^2 + 2a_1^2 k_1 x y_0 - 2a_1^2 k_1 x_0 y_0 - a_1^2 b_1^2 + a_1^2 y_0^2 + b_1^2 x^2 = 0, \quad (5)$$

and we collect the coefficients for x^2 , x and x^0 , and label them as A , B , and C respectively.

2. Because of the tangent line equations, we set the discriminant $B^2 - 4AC = 0$.
3. We collect the coefficients for k_1^2 , k_1 , and the constant term, k_1^0 respectively as follows:

$$\begin{aligned}d &= 4a_1^4b_1^2 - 4a_1^2b_1^2x_0^2, \\e &= 8a_1^2b_1^2x_0y_0, \\f &= 4a_1^2b_1^4 - 4a_1^2b_1^2y_0^2.\end{aligned}$$

4. We recall that if m_1 and m_2 are two slopes for two tangent lines, then we have

$$\begin{aligned}m_1 + m_2 &= -\frac{e}{d}, \\m_1m_2 &= \frac{f}{d}.\end{aligned}\tag{6}$$

Furthermore, the angle between these two lines, which we denote as α_1 , is

$$\tan \alpha_1 = \frac{m_2 - m_1}{1 + m_1m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2} = \frac{\sqrt{e^2 - 4df}}{d + f},\tag{7}$$

and hence

$$\alpha_1 = \tan^{-1} \left(\frac{\sqrt{e^2 - 4df}}{d + f} \right).$$

5. Analogously, we proceed to find the angle α_2 between two tangent lines for the second ellipse E_2 as follows

$$\alpha_2 = \tan^{-1} \left(\frac{\sqrt{e_1^2 - 4d_1f_1}}{d_1 + f_1} \right)$$

6. We recall that the orthoptic is the locus of points where two tangents to a given curve meet at a right angle. We note the Eq. (7), if $1 + m_1m_2 = 0$ or $d + f = 0$, then two tangent lines are perpendicular. In other words,

$$\frac{1}{a_1^2 - x_0^2} (a_1^2 + b_1^2 - x_0^2 - y_0^2) = 0,$$

and after substituting x_0 by x , and y_0 by y , we obtain

$$x^2 + y^2 = a_1^2 + b_1^2,\tag{8}$$

which is indeed the orthoptic for (1), and is known as the director circle. Similarly, we obtain the orthoptic for (2):

$$(x - c)^2 + (y - d)^2 = a_2^2 + b_2^2\tag{9}$$

7. We aim to solve for the equal viewpoint $\{x_0, y_0\}$ from

$$\alpha_1 - \alpha_2 = 0.\tag{10}$$

- (a) We first assume $\alpha_1, \alpha_2 \neq \frac{\pi}{2}$, then it can be shown that after substituting $x = x_0$, and $y = y_0$, the Eq. (10) is equivalent to

$$\begin{aligned} & \left[a_1^2 b_1^2 \left(\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1 \right) \right] \left[(a_2^2 + b_2^2 - ((x-c)^2 + (y-d)^2))^2 \right] \\ & - \left[(a_1^2 + b_1^2 - (x^2 + y^2))^2 \right] \left[a_2^2 b_2^2 \left(\frac{(x-c)^2}{a_2^2} + \frac{(y-d)^2}{b_2^2} - 1 \right) \right] = 0. \end{aligned} \quad (11)$$

- (b) In view of Eq. (6), the relationship between the slopes of two tangent lines. If $\alpha_1 = \frac{\pi}{2}$, then $\frac{f}{d} = -1$, which implies that $x_0^2 + y_0^2 = a_1^2 + b_1^2$. The locus, $x_0^2 + y_0^2 = a_1^2 + b_1^2$, is called the orthoptic for the ellipse $\mathbf{E}_1$ (1), also the director circle of the ellipse (1). Analogously, we may obtain the orthoptic for the ellipse (2). If we denote $\mathbf{Q} = a_1^2 + b_1^2 - x^2 - y^2$ to be the director circle (1), and $\mathbf{P} = a_2^2 + b_2^2 - ((x-c)^2 + (y-d)^2)$, then $Q = 0$ and $\mathbf{P} = 0$ are the director circles for $\mathbf{E}_1$ (1) and $\mathbf{E}_2$ (2) respectively. We consider

$$M_d = \begin{bmatrix} a_1^2 b_1^2 \left(\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1 \right) & a_2^2 b_2^2 \left(\frac{(x-c)^2}{a_2^2} + \frac{(y-d)^2}{b_2^2} - 1 \right) \\ (a_1^2 + b_1^2 - (x^2 + y^2))^2 & (a_2^2 + b_2^2 - ((x-c)^2 + (y-d)^2))^2 \end{bmatrix}, \quad (12)$$

then we see that $\det(M_d) = 0$ is

$$[a_1^2 b_1^2 \mathbf{E}_1] \cdot \mathbf{P}^2 - [a_2^2 b_2^2 \mathbf{E}_2] \cdot \mathbf{Q}^2 = 0, \quad (13)$$

which is equivalent to Eq. (11).

- (c) Common orthoptic for two ellipses can be found at [2].
 (d) We shall show that if $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, then the points belong to the equioptic curve for both (1) and (1) (see Theorem (2)).

If we expand Eq. (11), it becomes an equation of a degree 6 polynomial in two real variables x and y , which can be expressed as

$$P(x, y) = \sum_{i=0}^6 \sum_{j=0}^{6-i} a_{ij} x^i y^j = 0, \quad (14)$$

where a_{ij} are real constants. The plot of (11), an algebraic curve, could be complicated, involving sharp corners (or cusps). We call Eq. (11) to be a *quasi-equioptic* equation, and we denote the implicit plot of Eq. (11), the quasi-equioptic by L^Q . Since $\det(M_d) = 0$ is the equation for the quasi-equioptic for (1) and (2), and if we fix the ellipse (1), but vary the values of c or d for the ellipse (2), we invite readers to explore how a vertical or horizontal shifting on the second ellipse (2) affects their corresponding plots of quasi-equioptic respectively.

1.1 Quasi-equioptic when a rotated ellipse

Now, we maintain ellipse (1) invariant at the origin and apply a standard **clockwise rotation** of angle θ to ellipse (2) around its center. In other words,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

on (2). By substituting these mapping expressions back into the standardized unshifted conic template when $c = d = 0$, we successfully generate the clockwise-oriented profile:

$$\frac{(x \cos \theta - y \sin \theta)^2}{a_2^2} + \frac{(x \sin \theta + y \cos \theta)^2}{b_2^2} = 1. \quad (15)$$

Next, we substitute x by $x - c$ and y by $y - d$ to obtain the ellipse

$$\frac{((x - c) \cos \theta - (y - d) \sin \theta)^2}{a_2^2} + \frac{((x - c) \sin \theta + (y - d) \cos \theta)^2}{b_2^2} = 1. \quad (16)$$

We follow the preceding discussions to achieve the quasi-equioptic for (1) and (16) (by avoiding both the orthoptic curves for (1) and (16)), and we set

$$\det M_r = \begin{bmatrix} (1, 1) & (1, 2) \\ (2, 1) & (2, 2) \end{bmatrix} = 0, \quad (17)$$

where $(1, 1) = a_1^2 b_1^2 \left(\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1 \right)$,

$$(1, 2) = a_2^2 b_2^2 \left(\frac{((x - c) \cos \theta - (y - d) \sin \theta)^2}{a_2^2} + \frac{((x - c) \sin \theta + (y - d) \cos \theta)^2}{b_2^2} - 1 \right),$$

$$(2, 1) = (a_1^2 + b_1^2 - (x^2 + y^2))^2, \text{ and}$$

$$(2, 2) = (a_2^2 + b_2^2 - (((x - c) \cos \theta - (y - d) \sin \theta)^2 + ((x - c) \sin \theta + (y - d) \cos \theta)^2))^2.$$

2 The need to extract equioptic from a quasi-equioptic

Algebraically, we shall show that the final equioptic for the problem is extracted from a quasi-equioptic. In [1], we recall geometrically that we call a point $A = (x, y) \in \mathbb{R}^2$ is an equal viewpoint for two given smooth curves, and it belongs to a equioptic if A satisfies $\angle BAC = \angle DAE$ in Figure

2.

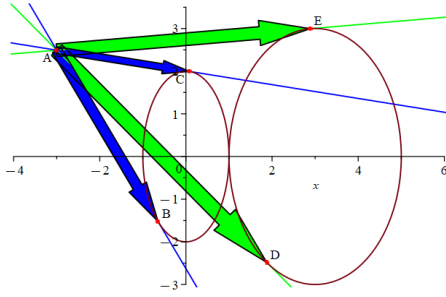


Figure 2. True equal viewpoint at the point A

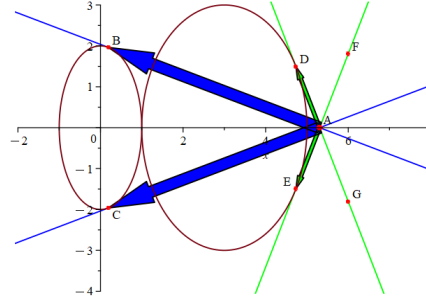


Figure 3. Quasi-equiopic at the point A

However, the points A we obtain algebraically from either (11) or (17) is satisfying either $\angle BAC = \angle DAE$ (see Figure 2), or its supplementary angle $\angle BAC = \pi - \angle DAE$ (see Figure 3). In both cases, we call A to be in the quasi-equiopic, and denote it by $A \in L^Q$. Our objective is to extract the quasi-equiopic L^Q further to form the equiopic for two curves. Simply put, if θ_1 is the angle between two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and if θ_2 is the angle two vectors $\overrightarrow{AD}$ and $\overrightarrow{AE}$, we need

$$\theta_1 - \theta_2 = 0, \quad (18)$$

Our approach is to use an algebraic approach to extract the equiopic from the quasi-equiopic that points in the equiopic will be equal viewpoints in $\mathbb{R}^2$ from a geometric point of view. To be precise, we do the following:

1. Derive an algebraic implicit equation for the quasi-equiopic as two variables x and y for two curves,
2. Solve the implicit equation in different branches by expressing y in terms of x .
3. Identify critical points for each branch, which can be the start or end point of an equiopic curve, a sharp corner, etc.
4. Use the fact that the quasi-equiopic is continuous in x and y in an interval (see Theorem (4), and
5. Apply numerical approaches to extract the equiopic by verifying if Eq. (18) is satisfied.

We see that if $A \in L^Q$, then either A is an equiopic or A is not. We, therefore, use L^T to denote the equiopic points and L^N to denote those that are not equiopic points. Subsequently, we see

$$L^Q = L^T \cup L^N.$$

The following is a simple observation from factorization using a computer algebra system, whose proof we omit.

Theorem 1 We consider Question (). If the ellipses (1) and (16), are two equal-sized ellipses with $c > 0$, and $d = 0$. Then the quasi-equiopic equation can be expressed as $c(c - 2x) \cdot p(x, y) = 0$, where $p(x, y)$ is a degree 4 polynomial of x and y .

In view of Figure 3, we see that the quasi-equiopic A is the set of points that satisfies either $\angle BAC = \angle DAF$ or $\angle BAC = \angle DAE$, and we note that $\angle DAF$ and $\angle DAE$ are supplementary angles of each other. They will not be the same unless they are $\frac{\pi}{2}$, or one of them is 0, in which case, the quasi-equiopic is a straight line.

Theorem 2 We consider the equiopic for the ellipses (1) and (2), see Figure 3. If $\angle BAC = \angle DAE = \angle DAF = \frac{\pi}{2}$, then the real solutions lie on the intersections of two respective director circles,

$$\begin{cases} (a_1^2 + b_1^2 - (x^2 + y^2)) = 0, \\ (a_2^2 + b_2^2 - ((x - c)^2 + (y - d)^2)) = 0 \end{cases} \quad (19)$$

are the intersecting points $L^T \cap L^N$. The intersections are both orthoptic points for the ellipses (1) and (2).

Proof: We refer to Figure 3 and assume $\angle BAC = \angle DAE = \angle DAF = \frac{\pi}{2}$, then A can be either an equal viewpoint or not an equal viewpoint. We proceed and set $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, and since $\tan \frac{\pi}{2}$ is undefined, we obtain $(a_1^2 + b_1^2 - (x^2 + y^2)) = (a_2^2 + b_2^2 - ((x - c)^2 + (y - d)^2)) = 0$, and it follows from Eq. (11) that when $\alpha_1 = \alpha_2 = \frac{\pi}{2}$, point A lies on the intersections of Eq. (19). ■

Since director circles are invariant under rotations, we naturally have the following observation.

Corollary 3 We consider the equiopic for the ellipses (1) and (16). If $\angle BAC = \angle DAE = \angle DAF = \frac{\pi}{2}$, then the real solutions of the intersections

$$\begin{cases} (a_1^2 + b_1^2 - (x^2 + y^2)) = 0, \\ a_2^2 + b_2^2 - (((x - c) \cos \theta - (y - d) \sin \theta)^2 + ((x - c) \sin \theta + (y - d) \cos \theta)^2) = 0 \end{cases} \quad (20)$$

are the same as in Eq. (19), and these are the orthoptic for both ellipses (1) and (16).

Remarks: Since the implicit plot of Eq. (11) or Eq. (17), with variables in x and y and other parameters are fixed, is continuous throughout an interval, we have the following observations:

1. Suppose the quasi-equiopic (11) or (17) is expressed in branches of functions, $\{A_i(x)\}_{i=1}^n$, and if $A_i(x)$ is a real branch, non-differentiable at $x = x^*$, and x^* is an equal viewpoint, then x^* is either a start or an endpoint for $A_i(x)$.
2. If $x = x^*$ is a start point and an equal viewpoint for $A_i(x)$, then $A_i(x)$ has equal viewpoints in some interval $[x^*, x^* + \alpha)$ for some $\alpha > 0$. If $x = x^*$ is an endpoint and an equal viewpoint for $A_i(x)$, then $A_i(x)$ has equal viewpoints in some interval $(x^* - \alpha, x^*]$ for some $\alpha > 0$. On the other hand, if $x = x^*$ is not an equal viewpoint for $A_i(x)$, then $A_i(x)$ does not have equal viewpoints in some interval containing x^* . To summarize, we have the following trivial observation.

Theorem 4 We consider the locus posted in the Question for the pair of two ellipses (1) and (2), or (1) and (16), when A is outside both ellipses. Then the plot of the equiopic or equiopic L^T is a continuous curve in a certain interval.

2.1 Examples

Example 5 We recall that when we use $a_1 = 1, b_1 = 2, a_2 = 2, b_2 = 3, c = 3, d = 0$ and $\theta = 0$ for (1) and (16), the equioptic curve for (1) and (2) has been discussed in [1], and the interactive file can be found in [S1], the final equioptic curve in this case is depicted in Figure 4.

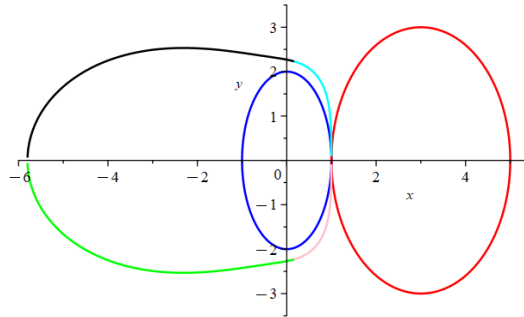


Figure 4. Equioptic curve for Example 5

Now we fix ellipse centered at the origin (1), and rotate the (2) by an angle θ .

Example 6 We use the same parameters for $a_i, b_i, i = 1, 2, c$, and d but choose $\theta = \frac{\pi}{4}$ for (1) and (16) as stated in Example (5), and we call these ellipses E_1 and E_2 , respectively. We include all detailed computations in [S2] but we summarize as follows:

1. We set $\alpha_1 - \alpha_2 = 0$ to obtain a polynomial degree 6 of x and y , which we recall that the implicit is called the quasi-equioptic L^Q .
2. We express y in terms of x in the degree 6 from L^Q to obtain at most 6 functions by using the following commands using [4]. First, we use ‘prelocus’ for quasi-equioptic to represent the Eq. (17) below:

```
AA:=solve(prelocus,y);
BB:=allvalues(AA);
bb1:=unapply(BB[1],x);
bb2:=unapply(BB[2],x);
bb3:=unapply(BB[3],x);
bb4:=unapply(BB[4],x);
bb5:=unapply(BB[5],x);
bb6:=unapply(BB[6],x);
```

3. We save the plot for each branch as follows:

```
branch1:=plot(bb1(x),view = [-14 .. 6, -8 .. 3]);
branch2:=plot(bb2(x),view = [-14 .. 6, -8 .. 3]);
:
branch6:=plot(bb6(x),view = [-14 .. 6, -8 .. 3]);
```

4. We display all 6 branches in the following color order of red, blue, pink, cyan, green, and black for branch1 through branch 6, respectively below, and note that branch 6 yields a complex plot, which is not shown.

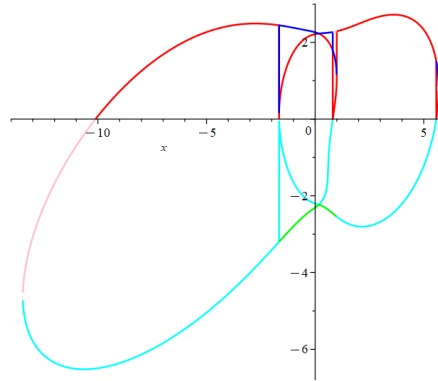


Figure 5. Plot of all branches in
Example 6

2.2 Procedure for forming an equioptic from a quasi-equioptic

In view of Eq. (18), to say $A \in L^T$, we need to have $\theta_1 - \theta_2 = 0$ or scientifically close to 0 after ignoring insignificant decimal. Since Eq. (11) or (17) is a continuous implicit curve of variables x and y , it follows from Theorem (4) if a point $A \in L^T$, then A should travel along a continuous curve; same thing can be said if $A \in L^N$. Therefore, if we start with $A \in L^Q$, we need to check computationally and verify if $A \in L^T$ or $A \in L^N$. However, when (14) is expressed as branches of functions, we shall see that individual branches of functions may not be a smooth curve. As a result, we call a sharp corner A or a cusp of a non-smooth curve a *turning point*, since A may be a point turning from L^T to L^N or vice versa. Furthermore, if $A \in L^T \cap L^N$, the Theorem (2) helps to determine if a point A is a starting, turning point, or an ending point for L^T or L^N . Therefore, we call A a *critical point* for a branch if it is a beginning, ending, or a turning point. We remark that the computational process of finding the critical points involves serious numerical computation; we thus set the `Digits:=15` when we use the computer algebra system [4]. Secondly, we need to take special care of the point $(1, 0)$, where both ellipses share a vertical tangent line, and thus the slope is undefined. However, it is an equal viewpoint, obviously. We describe how we proceed in the following steps:

1. Once the positions of two ellipses are determined. We use Theorem (2) to find intersection points $A \in L^T \cap L^N$.
2. We identify the critical points for a branch, and proceed with a point $A = (x_0, y_0)$ in a branch of the quasi-equioptic plot L .
3. We solve Eq. (5) for the slopes of the tangent lines for the ellipse (1) and call them k_{11} and k_{12} respectively.
4. We find the respective tangent line equations for k_{11} and k_{12} , respectively, and call them l_{11} and l_{12} respectively.

5. We find the tangent lines l_{21} and l_{22} respectively for the ellipse (16).
6. We next find the tangencies for these two respective ellipses, (1) and (16).
 - (a) We solve the intersections between $l_{1j}, j = 1, 2$, and ellipse (1) respectively, and label them (x_{11}, y_{11}) and (x_{12}, y_{12}) respectively.
 - (b) We do the same and find the intersections between $l_{2j}, j = 1, 2$, and ellipse (16) respectively, and label them (x_{21}, y_{21}) and (x_{22}, y_{22}) respectively.
 - (c) We form the vector

$$v_{ij} = (x_{ij} - x_0, y_{ij} - y_0),$$
 where $i, j = 1, 2$.
 - (d) We verify if the angle, α_1 , between v_{11} and v_{12} is the same as the angle, α_2 , between v_{21} and v_{22} up to some significant decimal places. If the pairs of angles are the same, then $A \in L^T$; otherwise $A \in L^N$. We write a program to speed up checking all points x in an interval $[a, b]$ to be an equal viewpoint or not.
7. When checking if a point A is an equal viewpoint. It is possible to run into A possesses either a horizontal or vertical tangent. In such a case, we shall manually select l_{ij} , for $i = 1, 2$, before verifying if A is an equal viewpoint. Furthermore, it is also possible that when A is a common orthoptic point for both ellipses, where $\alpha_1 = \alpha_2 = \frac{\pi}{2}$; subsequently, we could lose the accuracy of computation when computing $\alpha_1 - \alpha_2$.
8. To find the critical point for each branch, we always start with the x -coordinate from the left to the right.

2.2.1 Branch 1 and branch 2

We first examine the plots of branch1 and branch2, and find the critical points for $bb1(x)$ and $bb2(x)$.

1. In view of the Theorem (2), we find the intersection between branch1 and branch2 at $P_1 = \left(\frac{1}{6}, \frac{\sqrt{179}}{6}\right)$, and the intersection between branch4 and branch5 at $P_2 = \left(\frac{1}{6}, \frac{-\sqrt{179}}{6}\right)$. In addition, we see that P_1 and P_2 are also two common orthoptic for both ellipses (1) and (16). Therefore, P_1 and P_2 belong to the equioptic for the two given ellipses.
2. By setting $bb1(x)=0$, we find the starting point for branch1 is at $x_{11} = -10.11282032$.
3. The points of non-differentiability for $bb1(x)$ are at $x_{1c} = -1.660457342$, $x_{2c} = 0.1666666667$, and $x_{3c} = 0.9953159706$. We recall that $x_{2c} = \frac{1}{6} = 0.1666666667$ is an orthoptic for both ellipses, and the intersection between branch1 and branch2 is also at $x_{2c} = \frac{1}{6}$.
4. We make use of an animation program (see [S3]) written in [4] to visualize points throughout the interval $[x_{11}, x_{1c}]$ that are equioptic for $bb1(x)$. It is worth noting the following:

```
an:=i->TestprocAnim(aa+bb*i/30,Re(evalf(bb1(aa+bb*i/30))));
plots-display(seq(an(i),i=0..30),insequence=true);
```

- (a) We remark that $aa := x_{11}; bb := (x_{1c} - aa)$; where aa is the start point, bb is the length between aa and x_{1c} .
- (b) We note that when $i = 0$ is used, we are verifying if the start point is in equioptic; similarly, when $i = 30$ is used, we are verifying if the endpoint is in equioptic.
5. We make use of the animation and refer readers to [S3] that branch 2 consists of points in equioptic in $(x_{1c}, x_{2c}) = (-1.660457342, 0.1666666667)$.
6. We next find that the intersection points between the two given ellipses are at $[0.5691282360, -1.644497548]$ and $[0.9517756384, 0.6135898873]$ respectively. We label $x_{int1} = 0.9517756384$; next, we denote $x_{int2} = 0.5691282360$. Moreover, we the the second $x - intercept = 0.8118980591$ for $bb1(x)$ is denoted by x_{12} .
- (a) We zoom in the plots of branch1, branch2, (1), (16) for $x \in [x_{2c}, x_{int1}]$ (see Figure 6(a)), and together with P_1, P_2 , first and the second intersections, and the second $x - intercept$ for $bb1(x)$, x_{12} , see Figure 6(b).

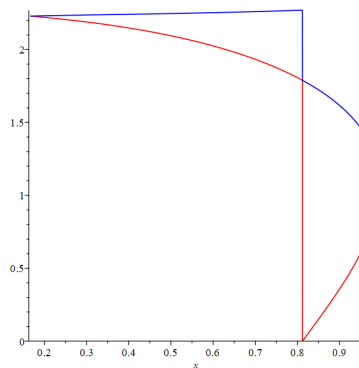


Figure 6(a). Zoomed in for branches 1 and 2

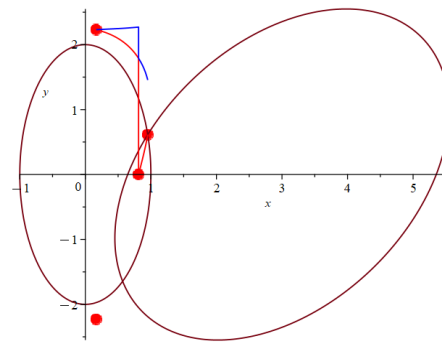


Figure 6(b). Zoomed out for branches 1 and 2

- (b) We verified that $bb1(x)$ has equioptic in the interval $[\frac{1}{6}, x_{12}]$.
- (c) We verified that $bb2(x)$ has equioptic in the interval $[x_{12}, x_{int1}]$.
- (d) We note that there is a vertical tangent for $bb1(x)$ near x_{int1} , we may find such a critical point by using `fsolve(denom(diff(bb1(x), x))=0, x=0.8..1);`
- (e) We find such a critical point to be the third cusp at $x_{3c} = 0.9953159706$.
- (f) We verified that $bb1(x)$ has equioptic in the interval

$$(x_{int1}, x_{3c}] = (0.951775637587971, 0.9953159706].$$

2.2.2 Branch 3, branch 4 and branch 5

1. We first depict $bb3(x)$ (in pink) and $bb4(x)$ (in cyan) in Figure 7:

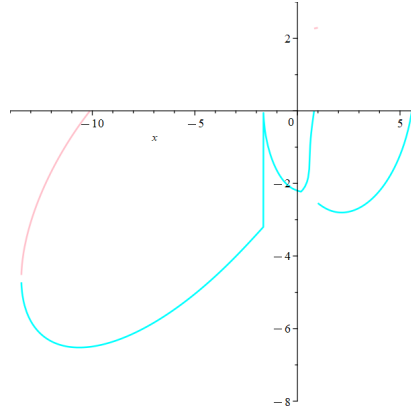


Figure 7. Plots of $bb3(x)$ and $bb4(x)$

2. We find the intersecting point between $bb3(x)$ and $bb4(x)$ at $x_{34} = -13.45742183$.
3. We verify that $bb3(x)$ has equioptic in $(x_{34}, x_{11}] = (-13.45742183, -10.11282032]$.
4. We leave it as an exercise to the readers (and see [S3]) that $bb4(x)$ has a discontinuity at its intercept $x_{4int} = -1.658899324$.
5. We verify that $bb4(x)$ has equioptic in $[x_{34}, x_{4int}) = [-13.45742183, -1.658899324)$.
6. We verify that $bb5(x)$ (in green) has equioptic in $[x_{4int}, \frac{1}{6}] = [-1.658899324, 0.166666667]$.
7. We verify that $bb4(x)$ has equioptic in $(\frac{1}{6}, x_{int2}) = (0.166666667, 0.5691282360)$.
8. Finally, we remind readers that the intersections between two ellipses, at $x = x_{int1}$, and x_{int2} , are equal viewpoints.
9. After collecting all working branches as equioptic, we depict the plot of those points that are

equioptic in Figure 8:

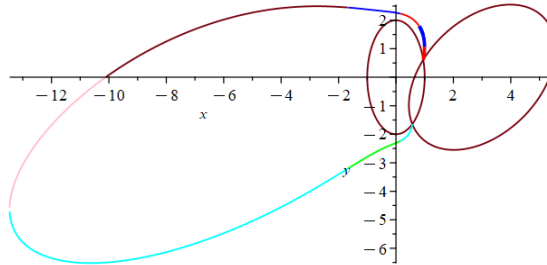


Figure 8. Equioptic for (1) and (16)

3 Uniform scaling on a locus

A uniform scaling is a linear transformation where all dimensions of a shape are scaled by the same factor. In [1], we proved that if the compressions or expansions are applied on both (1) and (2) with the same rate in both x and y directions, then we can adjust the locus Eq. (11) (with the same rate in both x and y directions) accordingly without calculating the new locus for two respective new equations. First, the quasi-equioptic equation (11) is valid under shifting and rotations. The analogous result can be stated for (1) and (16), and the proof is trivial, which we state below but omit its proof.

Theorem 7 Suppose $a, b \neq 0$, and consider two ellipses after we substitute x by $a \cdot x$ and y by $b \cdot y$ in both (1) and (16). We then compute its quasi-equioptic equation directly, and call it L_{ab}^1 . On the other hand, we substitute $x = a \cdot x$ and $y = b \cdot y$ directly in the quasi-equioptic Eq. (17), and call it L_{ab}^2 . Then we see $L_{ab}^1 \neq L_{ab}^2$. However, if we choose $a = b \neq 0$, then we have

$$\frac{L_{aa}^1}{a^8} = L_{aa}^2.$$

Theorem 8 We consider the locus in Question (). Let the ellipses (1) and (2) be two equal-sized ellipses, with $c > 0$, and $d = 0$. (i) If the ellipses (1) and (2) are disconnected, then the locus is $x = \frac{c}{2}$. (ii) If the ellipses (1) and (2) have intersections, then the locus will be the vertical line $x = \frac{c}{2}$ minus the line segment connecting two intersections.

Proof: For Part (i): We proved in the Theorem (1) that if we set $a_1 = a_2, b_1 = b_2$, and $d = 0$. Then the pre-locus equation (11) is

$$c(c - 2x) \cdot p(x, y) = 0,$$

where $p(x, y)$ is a degree 4 polynomial of x and y . In view of the Theorem (7), to find the equioptic curve in this case, without loss of generality, we may assume $a_1 = a_2 = 2, b_1 = b_2 = 1, c = 5$, and $d = 0$. The quasi-equioptic is shown in Figure 9(a). The ellipses (1) and (2) are shown in red and blue together with the quasi-equioptic in Figure 9(b). We see that $x = \frac{c}{2}$ is a perpendicular bisector of line

segment connecting $(a_1, 0)$ and $(c - a_2, 0)$. It follows from the definition that an equal viewpoint A should satisfy Eq. (18), we conclude that the points on $x = \frac{c}{2}$ are the only equioptic curve point in this case.

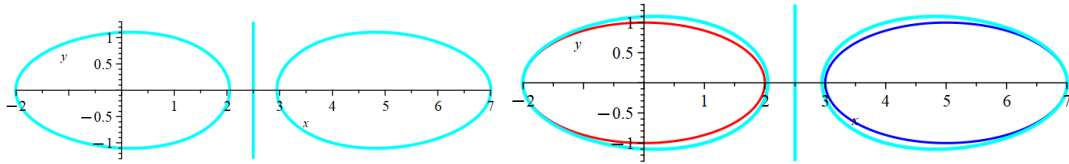


Figure 9(a). The quasi-equioptic curve in Part (i)

Figure 9(b). The quasi-equioptic together with two ellipses in Part (i)

For Part (ii): Without loss of generality, we assume $a_1 = a_2 = 2$, $b_1 = b_2 = 2$, $c = 3$, and $d = 0$. The pre-locus is shown in Figure 10(a). The ellipses (1) and (2) are shown in red and blue together with the pre-locus in Figure 10(b). We observe that points that are inside the intersections of two ellipses cannot be equal viewpoints by definition, and observations from Part (i) that points on $x = \frac{c}{2}$, which are outside the line segments connecting two intersecting points, are the only equioptic curve points in this case.

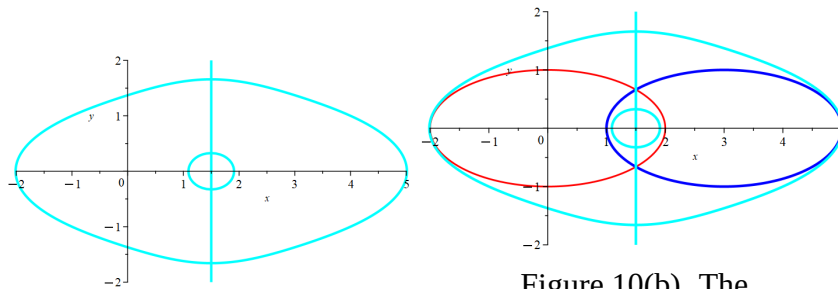


Figure 10(a). The quasi-equioptic curve in Part (ii)

Figure 10(b). The quasi-equioptic together with two ellipses in Part (ii)

3.1 Exercises

1. We consider the case of two non-intersecting and different-sized circles. For example, we consider the unit circle C_1 of $x^2 + y^2 - 1 = 0$, and C_2 of $(x - 4)^2 + y^2 = 4$. We follow the procedure in this paper to find that the equioptic is a circle of center $(-\frac{4}{3}, 0)$ and radius of $\frac{8}{3}$.
2. We consider two intersecting circles with different radii of circles. For example, we consider the unit circle C_1 of $x^2 + y^2 - 1 = 0$, and C_2 of $(x - 2)^2 + y^2 = 4$. The quasi-equioptic (in cyan) and respective circles (in red and blue respectively) for this problem are sketched in Figure 11(a). Readers should be able to sketch and conjecture the equioptic from the quasi-equioptic after

discussions from this paper. [Hint: See the equioptic shown in dark green in Figure 11(b).]

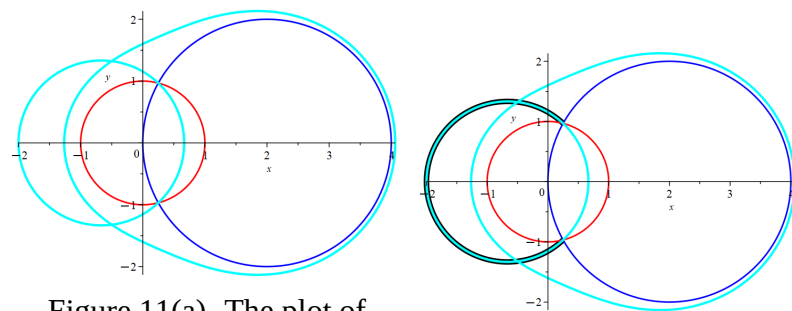


Figure 11(a). The plot of quasi-locus in cyan and two respective circles

Figure 11(b) True locus shown in dark green

3.2 Transformations on quasi-equioptic for two ellipses, and a conjecture

Given two ellipses, without loss of generality, we assume the first ellipse is in the form of (1) and the second ellipse is in the form of (16) after a proper rotation and shiftings. In view of the Theorem (7), once two ellipses (1) and (16) are chosen, the quasi-equioptic (17) is determined up to a uniform scaling, and hence its true equioptic curve can be determined up to a uniform scaling as well. We recall that a rigid transformation is geometric transformations that preserve distances and angles. It is clear that if f is a rigid transformation, and the equioptic Q_{12} is found for E_1 (1) and E_2 (16), then $f(Q_{12})$ is identically the equioptic for $f(E_1)$ and $f(E_2)$. In other words, the geometric operation of finding the equioptic locus commutes with rigid transformations:

$$Q(f(E_1), f(E_2)) = f(Q(E_1, E_2)) = f(Q_{12}).$$

In this subsection, we shall outline how we come up with the Conjecture (3.2). First, we express the quasi-equioptic for $E_1 \cup E_2$ in various real branches first according to the quasi-equioptic equation (17). Although we cannot obtain the equioptic curves for $E_1 \cup E_2$ after a horizontal, vertical shifting or rotating E_2 alone; however, the quasi-equioptic equation (17) is still useful. In particular, we shall observe how a horizontal or vertical shifting would affect the real branches of the quasi-equioptic when we vary the corresponding parameters c, d and θ . As expected, some real branches may not exist when we vary a particular parameter, c, d or θ .

Example 9 We substitute all parameters used in Example (5) except keeping d as a variable in quasi-equioptic equation (11).

We use the animations (see [S3]) to see how a vertical shifting, due to the variable d , affects the corresponding quasi-equioptic. We depict the quasi-equioptic (in red), ellipse (1 in green) and ellipse

(2 in blue) when $d = -3, -1.5$ and 3 respectively in Figures 13(a)-(c).

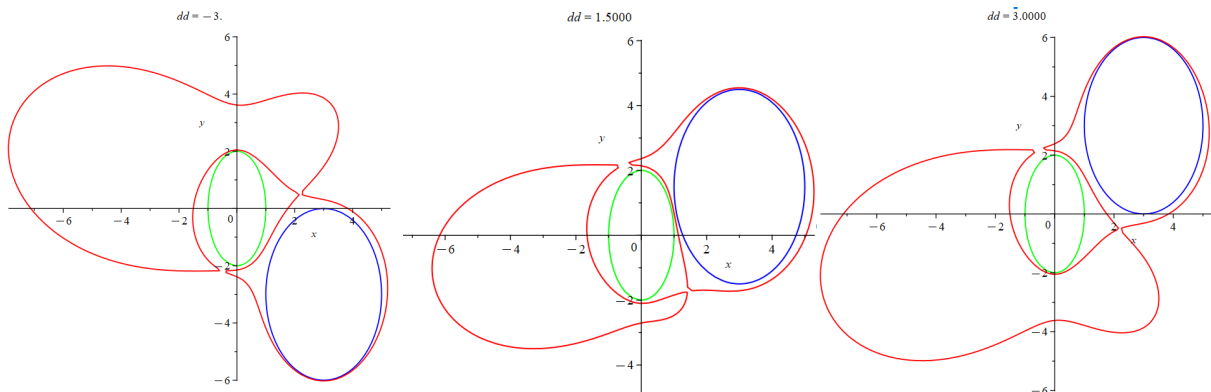


Figure 13(a) Quasi-equiopptic when $d = -3$

Figure 13(b) Quasi-equiopptic when $d = -1.5$

Figure 13(c) Quasi-equiopptic when $d = 3$

Furthermore, we can express (11) as various branches, as variables x and d , and see how various branches behave after a vertical shifting.

Example 10 We substitute all parameters used in Example (5) except keeping c as a variable in quasi-equiopptic equation (11).

We use [S4] to demonstrate how a horizontal shifting, due to the variable c , affects the corresponding quasi-equiopptic. We depict the quasi-equiopptic (in red), ellipse (1 in green) and ellipse (2 in blue) when $c = -4, -1$ and $c = 2$, respectively, in Figures 14(a)-(c).

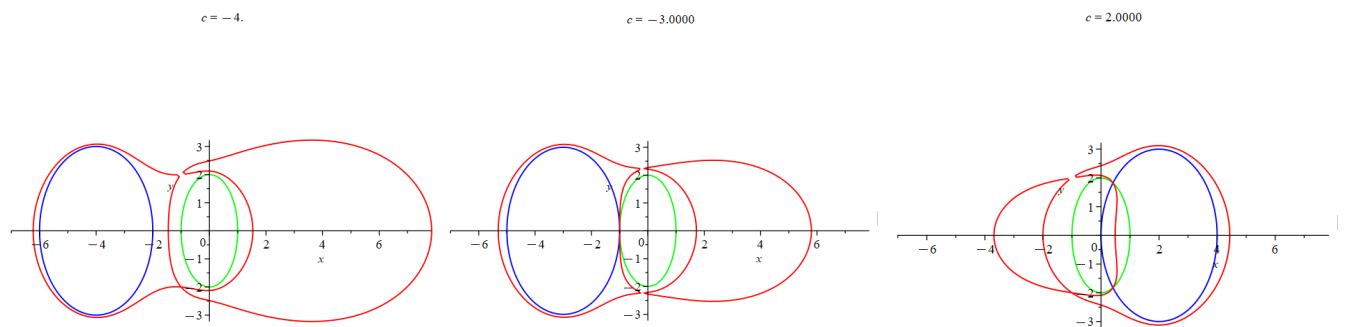


Figure 14(a) Quasi-equiopptic when $c = -4$

Figure 14(b) Quasi-equiopptic when $c = -1$

Figure 14(c) Quasi-equiopptic when $c = 2$

Furthermore, we can express (11) as various branches, as variables x and c , and see how various branches behave after a horizontal shifting.

Example 11 We substitute all parameters used in Example (6) except keeping θ as a variable in the quasi-equiopptic equation (17).

We see in [S5] to demonstrate how a rotation angle θ affect on the corresponding quasi-equioptic. We depict the quasi-equioptic (in red), ellipse (1 in green) and ellipse (2 in blue) when $\theta = -\frac{\pi}{4} \approx -0.78540$, -0.45815 and $\theta = 0.65450$, respectively, in Figures 15(a)-(c).

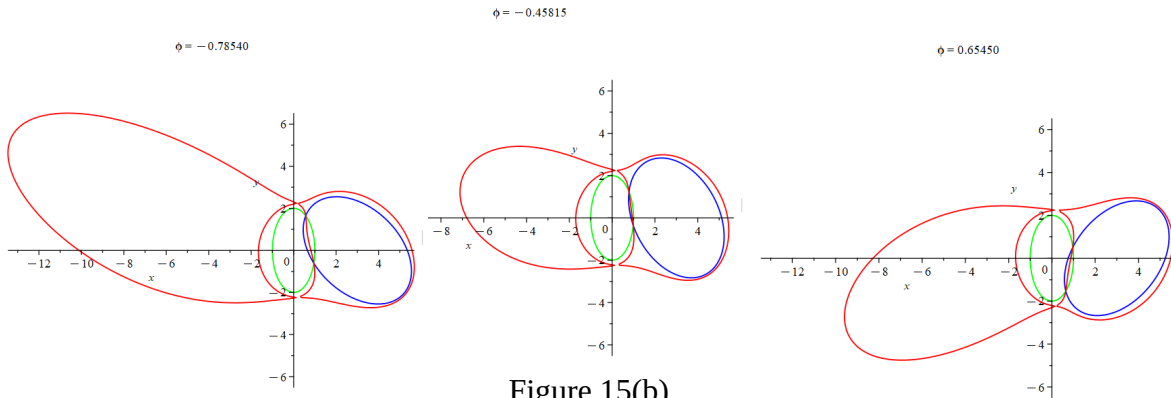


Figure 15(a) Quasi-equioptic when $\theta = -\frac{\pi}{4}$

Figure 15(b) Quasi-equioptic when $\theta = -0.45815$

Figure 15(c) Quasi-equioptic when $\theta = 0.65450$

We finish this section by making the following

Conjecture : If we consider two ellipses in the form of $E_1 : \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$ and $E_2^* : \frac{((x-c) \cos \theta - (y-d) \sin \theta)^2}{a_2^2} + \frac{((x-c) \sin \theta + (y-d) \cos \theta)^2}{b_2^2} = 1$. If $\max(a_1, b_1) < \min(a_2, b_2)$, and E_1 and E_2^* are separated (i.e. $E_1 \not\subset E_2^*$), for any fixed parameters c, d and θ . Then we conjecture that the equioptic is the outer smooth closed curve enclosing the ellipse E_1 , (see [S6]).

4 Equal viewpoints for infinitely many circles or spheres

It is natural to explore an equioptic that will work for infinitely many circles. In Theorem (12), we see how the origin $(0, 0)$ is the equioptic for infinitely many shrinking circles within a given circle. Subsequently, we see how a line segment within a given circle is being equioptic for infinitely many equal-sized circles in Corollary (13). To start, we subdivide a circle into several equal circular sectors. We assume each circular sector of the circle is with the angle $\theta_m = \frac{2\pi}{m}$ starting from the origin, where m is an integer greater than or equal to 4.

Theorem 12 We let m be an integer greater than or equal to 4, and we consider the circle $C_0 : x^2 + y^2 = r^2$ and $\theta_m = \frac{2\pi}{m}$. Then we can fit infinitely circles C_n inside C_0 , such that for $i \in \mathbb{N}$, C_i and C_{i-1} are tangent to each other, and each C_i is contained in an evenly divided sector of angle θ_m of the circle C_0 , with radii,

$$d_i = \frac{r - 2 \sum_{j=1}^{i-1} d_j}{s + 1},$$

and center $\left(0, \frac{d_i}{\sin \theta_m}\right)$, where $s = \frac{1}{\sin \theta_m}$, and $d_1 = \frac{r}{1 + s}$. Furthermore, the center $(0, 0)$ is the equioptic or equal viewpoint for infinitely many circles.

Proof: We fix the even integer $m \geq 4$.

Step 1. We let the point $P_1 = (0, y_1) = \left(0, \frac{d_1}{\sin \theta_m}\right)$ to be the center of the circle C_1 , with $y_1 > 0$. We need the point $Q_1 = (q_{1x}, q_{1y})$ on line $l_1 : y = \tan\left(\frac{\pi}{2} - \theta_m\right)x$, so that R_1 is $(0, r)$, and $P_1R_1 = P_1Q_1$, and P_1Q_1 is perpendicular to l_1 . In other words, we consider the right triangle OP_1Q_1 with angle $\angle P_1OQ_1 = \theta_m$. If $\overline{P_1Q_1} = d_1$, then the hypotenuse $\overline{OP_1} = y_1 = sd_1$. Now, if the radius of C_1 is $\overline{P_1Q_1} = d_1$, then

$$\begin{aligned}\overline{P_1R_1} &= r - y_1 = r - sd_1 \\ &= \overline{P_1Q_1} = d_1 \\ \Rightarrow d_1 &= \frac{r}{s+1}.\end{aligned}$$

Therefore, C_1 is the circle with center $P_1 = (0, y_1) = (0, sd_1) = \left(0, \frac{d_1}{\sin \theta_m}\right)$, and radius $d_1 = \frac{r}{s+1}$. We refer to Figure 16, when we use $m = 12$, $\theta_m = \frac{\pi}{6}$, and $r = 2$ for C_0 (shown in red), and show how C_1 (shown in blue) is constructed with center P_1 , and radius $d_1 = \frac{2}{1+s}$, where $s = \frac{1}{\sin \frac{\pi}{6}}$.

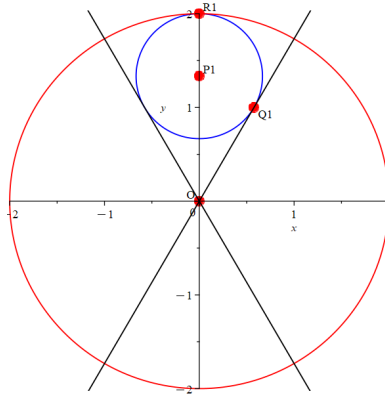


Figure 16. The circle C_1 , and center P_1

Step 2. Next, we continue to find the circle C_2 . We want to find the point $P_2 = (0, y_2)$, with $y_2 > 0$, and a point $Q_2 = (q_{2x}, q_{2y})$ on $y = \tan\left(\frac{\pi}{2} - \theta_m\right)x$, so that R_2 is $(0, r - 2d_1)$, and $P_2R_2 = P_2Q_2$, and P_2Q_2 is perpendicular to l_1 . We consider the right triangle OP_2Q_2 . If we set the radius $P_2Q_2 = d_2$, then the hypotenuse $\overline{OP_2} = y_2 = sd_2$, and we have

$$\begin{aligned}P_2R_2 &= (r - 2d_1) - y_2 = (r - 2d_1) - sd_2 \\ &= P_2Q_2 = d_2, \\ \Rightarrow d_2 &= \frac{r - 2d_1}{s+1} = \frac{r - 2\frac{2}{1+s}}{s+1} = \frac{\frac{r+rs-4}{1+s}}{s+1} = \frac{r(s-1)}{(s+1)^2}\end{aligned}$$

Thus, the circle is $x^2 + (y - y_2)^2 = (d_2)^2$.

Step 3. We continue this process to create the circles C_n with center $\left(0, \frac{d_n}{\sin \theta_m}\right)$, and radius $d_n = \frac{r - 2 \sum_{i=1}^{n-1} d_i}{s + 1} = \frac{(r - 2d_1)(s - 1)^{n-2}}{(s + 1)^{n-1}}$, $n = 1, 2, \dots$. Through the constructions of the circles, C_i , we see that C_i and C_{i-1} are tangent to each other. Since $2d_i$ is the diameter of C_i , $i \in \mathbb{N}$, and C_1 is tangent to C_0 , we get,

$$\begin{aligned} & \sum_{i=1}^{\infty} 2d_i \\ &= 2 \left(d_1 + \frac{r - 2d_1}{s + 1} + (r - 2d_1) \frac{s - 1}{(s + 1)^2} + (r - 2d_1) \frac{(s - 1)^2}{(s + 1)^3} + \dots + (r - 2d_1) \frac{(s - 1)^{n-2}}{(s + 1)^{n-1}} + \dots \right) \\ &= 2 \left[d_1 + \frac{r - 2d_1}{s + 1} \left(1 + \left(\frac{s - 1}{s + 1} \right) + \left(\frac{s - 1}{s + 1} \right)^2 + \dots + \left(\frac{s - 1}{s + 1} \right)^{n-2} \right) + \dots \right] \\ &= 2 \left[d_1 + \frac{r - 2d_1}{s + 1} \left(\frac{1}{1 - \left(\frac{s - 1}{s + 1} \right)} \right) \right], \text{ (since } s = \frac{1}{\sin \theta_m}, \text{ and } \frac{s - 1}{s + 1} = \frac{1 - \sin \theta_m}{\sin \theta_m + 1} < 1) \\ &= 2 \left[d_1 + \frac{r - 2d_1}{s + 1} \left(\frac{1}{\frac{1}{2}s + \frac{1}{2}} \right) \right] = 2 \left[d_1 + \left(\frac{r - 2d_1}{2} \right) \right] = 2d_1 + r - 2d_1 = r. \end{aligned}$$

Thus we can fit infinitely many circles C_i in each circular sector, and it is trivial that $(0, 0)$ is the equal viewpoint for infinitely many circles C_i , $i \in \mathbb{N}$, which are bounded by two lines $y = \tan\left(\frac{\pi}{2} - \theta_m\right)$, and $y = \tan\left(\frac{\pi}{2} + \theta_m\right)$.

Corollary 13 In Theorem (12), if m is an **even** integer greater than or equal to 4. Then (i) we have $\frac{m}{2}$ copies of congruent circles C_n , $n \in \mathbb{N}$, along the xy -plane. (ii) There are $\frac{m}{4}$ tangent lines that are equioptic or equal viewpoints for congruent copies of circles C_i , $i \in \mathbb{N}$ from adjacent circular sectors.

Proof: (i) Let $\theta_m = \frac{2\pi}{m}$, where m is an even integer greater than or equal to 4. We can divide a given circle C_0 , center at the origin O , into $\frac{m}{2}$ equal circular sectors ($\frac{2\pi}{2\theta_m} = \frac{\pi}{\theta_m} = \frac{m}{2}$). Since m is an even integer greater than or equal to 4, we first let T_1 and T_2 be two intersection points between $y = \tan\left(\frac{\pi}{2} - \theta_m\right)$, $y = \tan\left(\frac{\pi}{2} + \theta_m\right)$ and C_0 respectively, see Figure 17 when $m = 12$. Then it follows from Theorem (12) that we can construct infinitely many circles C_i , within OT_1T_2 , with $i \in \mathbb{N}$. We rotate the circular sector OT_1T_2 counter clock-wise to create $\frac{m}{2}$ congruent copies of circular sectors V_j , where $j = 1, 2, \dots, \frac{m}{2}$, $n \in \mathbb{N}$, within the given circle C_0 . We label these $\frac{m}{2}$ copies of pair of circular sectors of angle θ_m as $\{V_j, C_{j,n}\}$, and $C_{j,n}$ are congruent copies of circle $C_{1,n}$, with $C_{1,n} = C_n$, where $j = 1, 2, \dots, \frac{m}{2}$, $n \in \mathbb{N}$.

(ii) Since the circle C_0 is divided into $\frac{m}{2}$ even copies of circular sectors, $\{V_j, C_{j,n}\}$ where $j = 1, 2, \dots, \frac{m}{2}$, $n \in \mathbb{N}$. We label the circular sector that is symmetric to V_j , with respect to the origin O , as V_j^* . Hence both V_j and V_j^* are bounded two tangent lines of $y = \tan\left(\frac{\pi}{2} + (j - 2)\theta_m\right)x$ and $y = \tan\left(\frac{\pi}{2} + j\theta_m\right)x$. Therefore, we have $\frac{m}{4}$ tangent lines. For each $j = 1, 2, \dots, \frac{m}{2}$, the tangent line $y = \tan\left(\frac{\pi}{2} + j\theta_m\right)x$ is the equioptic or equal viewpoints for the congruent circles $C_{j,n}$ and $C_{(j+1),n}$,

with $n \in \mathbb{N}$.

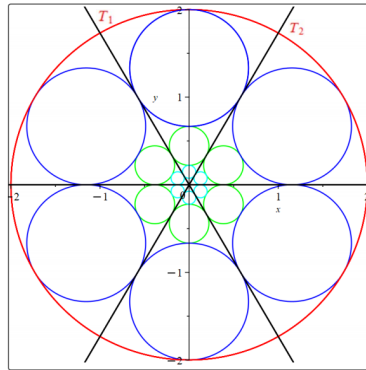


Figure 17. When $m = 12$

A Python code for generating Figure 17 can be found in [S8].

5 Filling a Cone with Non-Overlapping Spheres

We extend the 2D framework developed in Theorem (12) and Corollary (13) to 3D scenarios. We define point A as an equal viewpoint for two spheres S_1 and S_2 if there exist two right circular cones N_1 and N_2 sharing an identical half-angle θ with vertex A , such that each sphere S_i is tangent to its respective cone N_i ($i = 1, 2$). It is geometrically evident that the locus of equal viewpoints for two non-overlapping, equal-sized spheres forms a plane.

We begin by establishing an observation analogous to Theorem (12), providing a rigorous footing for infinite volumetric packings inside a bounding sphere.

Theorem 14 Let S_0 be a bounding sphere defined by $x^2 + y^2 + z^2 = r^2$, and let $\theta_m = \frac{2\pi}{m}$, where m is an integer such that $m \geq 4$. Define the scaling parameters $s = \frac{1}{\sin \theta_m}$ and $d_1 = \frac{r}{1+s}$. Then, one can embed an infinite sequence of non-overlapping spheres $\{S_n\}_{n=1}^{\infty}$ inside S_0 bounded by the conical surface $y = \tan\left(\frac{\pi}{2} - \theta_m\right) \sqrt{x^2 + z^2}$.

Consecutive spheres S_i and S_{i-1} are mutually tangent for $i \geq 2$, where each individual sphere S_n possesses a radius d_n given by:

$$d_n = \frac{r - 2 \sum_{i=1}^{n-1} d_i}{s + 1} = \frac{(r - 2d_1)(s - 1)^{n-2}}{(s + 1)^{n-1}}, \quad \forall n \geq 2, \quad (21)$$

and is centered at $\left(0, \frac{d_n + 2 \sum_{i=1}^{n-1} d_i}{\sin \theta_m}, 0\right)$. Furthermore, the origin $(0, 0, 0)$ is the unique equioptic point (equal viewpoint) for this infinite collection of shrinking spheres.

Proof. The proof proceeds by induction on the spatial constraints along the axis of symmetry (the y -axis), mapping the 3D tangential envelope directly to the plane section $z = 0$, thereby preserving the geometric proportions established in Theorem (12). Setting the vertex of the bounding cone at

the origin validates that the apparent angular diameter of every sphere S_n viewed from $(0, 0, 0)$ is identically $2\theta_m$. ■

We depict the scenario when $m = 12$ and 6 copies of cones of angle θ_m with $\{N_j, S_{j,n}\}$, $j = 1, 2, \dots, 6, n \in \mathbb{N}$ in Figures 18(a) and 18(b), see [S7] or [S8].

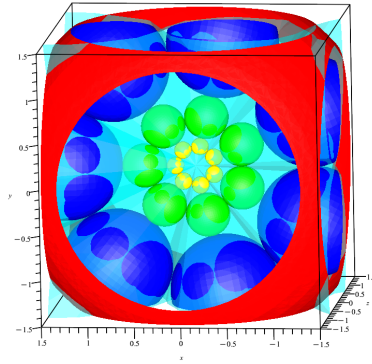


Figure 18(a) $m = 12$ and 6 copies of pair of cones

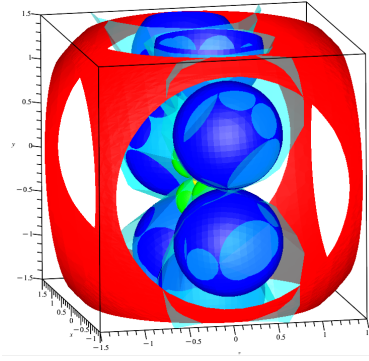


Figure 18(b) Another view of $m = 12$ and 6 copies of pair of cones

Corollary 15 *Subject to the conditions of Theorem 14, if m is an even integer satisfying $m \geq 4$, then:*

1. *There exist $\frac{m}{2}$ copies of congruent cones matching the profile of $y = \tan\left(\frac{\pi}{2} - \theta_m\right) \sqrt{x^2 + z^2}$, each containing an identical sequence of nested spheres $\{S_n\}_{n=1}^\infty$.*
2. *There exist exactly $\frac{m}{4}$ tangent planes intersecting at the origin that serve as continuous equioptic surfaces for congruent spheres from adjacent conical sectors.*

Proof. (i) Let $\theta_m = \frac{2\pi}{m}$. The total angular domain of the plane configuration is partitioned into $\frac{m}{2}$ symmetrical sub-domains via rotation about the z -axis. Let N_1 denote the primary spherical cone intersecting S_0 . Rotating N_1 tracking the transformation group $\mathbf{R}_z\left(\frac{4\pi}{m}\right)$ generates $\frac{m}{2}$ congruent, non-overlapping conical chambers N_j ($j = 1, 2, \dots, \frac{m}{2}$), each housing a sequence of congruent spheres $S_{j,n}$ such that $S_{1,n} \equiv S_n$.

(ii) For any fixed shell index $n \in \mathbb{N}$ and adjacent chamber indices $i \neq j$, the spheres $S_{i,n}$ and $S_{j,n}$ are isometric. The bounding interfaces separate adjacent symmetric cones along the planes defined by $y = \tan\left(\frac{\pi}{2} + j\theta_m\right) x$. Consequently, these interfacial boundaries represent $\frac{m}{4}$ distinct planes which serve as the true equioptic loci where the subtended viewing angles of adjacent spheres remain perfectly identical. ■

5.1 Analytical Upper Bounds and Spherical Cap Packing Density

To evaluate the global capacity of this configuration, we examine the maximum number of equal cones with half-angle θ_m that can share a singular vertex at the origin without mutual volumetric intersection. This configuration problem maps directly onto the surface S^2 as a spherical code problem.

Each right circular cone with a apex half-angle $\theta_m = \frac{\pi}{10}$ (corresponding to $m = 20$) projects an identical circular cap onto the unit sphere with an angular radius of $\alpha = \frac{\pi}{10}$. To ensure zero overlap between any two distinct cones, the minimal angular separation γ between their central axial vectors must strictly satisfy:

$$\gamma \geq 2\alpha = \frac{\pi}{5} = 36^\circ. \quad (22)$$

The maximum cardinality of this configuration represents the classical Tammes. Problem constraint for a minimum angular clearance of $\theta = 36^\circ$.

5.1.1 Theoretical Boundary via Area Allocation

The surface area of a single isolated spherical cap on a sphere of radius r is defined by:

$$\text{SA}_{\text{cap}} = 2\pi r^2(1 - \cos \alpha). \quad (23)$$

For a terminal radius $r = 2$ and an angular boundary $\alpha = 18^\circ$, the area equals $\text{SA}_{\text{cap}} = 8\pi(1 - \cos 18^\circ) \approx 1.23008$. Dividing the total surface area of the bounding sphere ($4\pi r^2 = 16\pi \approx 50.26548$) by SA_{cap} defines an absolute geometric upper bound:

$$N_{\max} \leq \left\lfloor \frac{4\pi r^2}{2\pi r^2(1 - \cos 18^\circ)} \right\rfloor = \left\lfloor \frac{2}{1 - \cos 18^\circ} \right\rfloor = 40. \quad (24)$$

5.1.2 Sharp Topological Bounds via Linear Programming

Although surface area conservation admits a theoretical maximum of 40 caps, empty interstitial voids are structurally compulsory when packing circles onto a curved manifold. To evaluate the exact configuration limit, we apply Delsarte's Linear Programming bounds for spherical codes.

By constructing an optimization polynomial $f(x)$ expanded via Gegenbauer polynomials $P_k(x)$ optimized for dimension $d = 3$, the objective function maintains $f(x) \leq 0$ for all inner products $x \in [-1, \cos 36^\circ]$. Solving this definitive linear programming envelope establishes that the maximum packing cardinality is strictly bounded by:

$$32 \leq N \leq 36. \quad (25)$$

This confirms that the area-based threshold of 40 can never be realized in physical space due to unavoidable packing gaps on the S^2 sphere topology.

5.2 Open Problems and Analytical Trajectories

While Corollary 15 outlines a rigid planar rotation model generating discrete sets of symmetric chambers, it isolates a constrained subset of global volumetric optimizations. We propose the following open question to steer subsequent research in geometric modeling and astronomical coordinates:

For an arbitrary, fixed cone half-angle θ_m , what is the absolute maximum cardinality $N_{\max}$ of non-overlapping, equal-sized cones sharing a singular vertex at the origin, and what topological transformation groups optimize the global packing density function ρ , defined by:

$$\rho(n, \theta_m) = \frac{n(1 - \cos \theta_m)}{2} ? \quad (26)$$

Investigating this problem requires calculating the boundary conditions where the asymptotic density deficiency $\Delta\rho = 1 - \rho_{\max}$ scales as $\theta_m \rightarrow 0$. Additionally, exploring configurations where the cone vertices are freed from the origin and allowed to slide along an inner core manifold may yield higher volumetric packing efficiencies. This provides valuable analytical utility for mapping multi-body orbital mechanics and optimizing deep-space telescopic tracking configurations.

6 Conclusions

We extract and analyze the branches of quasi-equioptic for two ellipses to form the final equioptic curve. The concept is simple; however, it is impossible to find the equioptic without the help of a computer algebra system. This locus problem is a strong example demonstrating why students need to be technologically competent so they can handle many real-life challenging problems. We observe from Google Search that many real-life applications can be found, which we leave it to readers to explore further.

Furthermore, as we see, we extended the equioptic problems for two ellipses to equal viewpoints for infinitely many circles in 2D within an evenly divided circle in a given circle. We also extended this to the corresponding 3D to visualize the equioptics for infinitely many spheres within an evenly divided cone in a given sphere. The author was surprised that the exploration leads to a complex 3D packing problem. Thus, the author believes that the equioptic curve plays an important role in computer science and geometric modeling fields. Finally, many interactive files can be found in the section of Supplementary Electronic Materials, and note that [S8] was generated by A.I.

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8 Supplementary Electronic Materials

[S1] Maple worksheet for Example 5.

[S2] Maple worksheet for Example 6.

[S3] Maple worksheet for Example 9.

[S4] Maple worksheet for Example 10.

[S5] Maple worksheet for Example 11.

[S6] Maple worksheet for the conjecture on two non-equal sized and disjoint ellipses.

[S7] Maple worksheet for six copies of cones.

[S8] Python code for generating Figures 17 and 18.

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